

ΣΧΕΣΕΙΣ

Εστω $f, g: A \rightarrow \mathbb{R}$, τότε:

i) $\inf(-f) = -\sup f$ και $\sup(-f) = -\inf f$

ii) $(\forall x \in A) : f(x) \leq g(x) \Rightarrow \begin{cases} \sup f \leq \sup g \\ \inf f \leq \inf g \end{cases}$

iii) $\sup(f+g) \leq \sup f + \sup g$

iv) $\sup(f+g) \geq \sup f + \inf g$

v) $\sup(f+c) = \sup f + c, c \in \mathbb{R}$

vi) $\sup(|c|f) = |c| \cdot \sup f, c \in \mathbb{R}$

vii) $\inf(f+g) \geq \inf f + \inf g$

viii) $\inf(f+g) \leq \inf f + \sup g$

ix) $\inf(f+c) = \inf f + c$

x) $\inf(|c|f) = |c| \cdot \inf f, c \in \mathbb{R}$

Απόδειξη

i) $(\forall x \in A) : f(x) \leq \sup f \Rightarrow -f(x) \geq -\sup f \Rightarrow -\sup f$ κ.φ $(-f)$

~~και~~ $\inf(-f) \geq -\sup f$ ①

$(\forall x \in A) : -f(x) = (-f)(x) \geq \inf(-f) \Rightarrow f(x) \leq -\inf(-f) \Rightarrow$
 $\Rightarrow -\inf(-f)$ ά.φ f

~~και~~ $\sup f \leq -\inf(-f) \Rightarrow \inf(-f) \leq -\sup f$ ②

Από ①+② έχουμε η (i)

ii) $(\forall x \in A) : \inf f \leq f(x) \Rightarrow -f(x) \leq -\inf f \Rightarrow -\inf f$ ά.φ $(-f)$

~~και~~ $\sup(-f) \geq -\inf f$ ①'

$(\forall x \in A) : -f(x) = (-f)(x) \leq \sup(-f) \Rightarrow f(x) \geq -\sup(-f) \Rightarrow$
 $\Rightarrow -\sup(-f)$ κ.φ f

~~και~~ $\inf f \geq -\sup(-f) \Rightarrow -\inf f \leq \sup(-f)$ ②'

Από ①'+②' έχουμε η (ii)

ii) $(\forall x \in A) : g(x) \leq \text{Sup } g \Rightarrow f(x) \leq \text{Sup } g \Rightarrow \text{Sup } g \text{ a.p } f$
 $\text{a.p } g, \text{ Sup } f \leq \text{Sup } g.$

$(\forall x \in A) : \text{Inf } f \leq f(x) \Rightarrow \text{Inf } f \leq g(x) \Rightarrow \text{Inf } f \text{ k.p } g$
 $\text{a.p } g, \text{ Inf } f \leq \text{Inf } g.$

iii) $(\forall x \in A) : \begin{cases} f(x) \leq \text{Sup } f \\ g(x) \leq \text{Sup } g \end{cases} \Rightarrow f(x) + g(x) \leq \text{Sup } f + \text{Sup } g \Rightarrow$
 $\Rightarrow (\text{Sup } f + \text{Sup } g) \text{ a.p } \text{ur } (f+g).$

$\text{a.p } g, \text{ Sup } (f+g) \leq \text{Sup } f + \text{Sup } g$

iv) $(\forall x \in A) : \text{Sup } f = \text{Sup}((f+g)-g) \stackrel{(iii)}{\leq} \text{Sup}(f+g) + \text{Sup}(-g) \stackrel{(i)}{=} \\ = \text{Sup}(f+g) - \text{Inf } g \Rightarrow \text{Sup } f = \text{Sup}(f+g) - \text{Inf } g \Rightarrow \\ \Rightarrow \text{Sup}(f+g) = \text{Sup } f + \text{Inf } g.$

v) Anò (iii) $\text{Sup}(f+c) \leq \text{Sup } f + \text{Sup } c = \text{Sup } f + c$

Anò (iv) $\text{Sup}(f+c) \geq \text{Sup } f + \text{Inf } c = \text{Sup } f + c$

A.p.a l'axioma (v)

vi) Γ a $c=0$ l'axioma.

Γ a $c \neq 0, (\forall x \in A) : |c| \cdot f(x) \leq |c| \cdot \text{Sup } f \Rightarrow$

$\Rightarrow (|c| \cdot f)(x) \leq |c| \cdot \text{Sup } f \Rightarrow |c| \cdot \text{Sup } f \text{ a.p } \text{ur } (|c| \cdot f)$
 $\text{a.p } g, \text{ Sup}(|c| \cdot f) \leq |c| \cdot \text{Sup } f \quad (*)$

Enò, $f = \frac{1}{|c|} |c| \cdot f \Rightarrow \text{Sup } f = \text{Sup} \left(\frac{1}{|c|} |c| \cdot f \right) = \\ = \text{Sup} \left(\frac{1}{|c|} (|c| \cdot f) \right) \stackrel{(*)}{\leq} \frac{1}{|c|} \cdot \text{Sup}(|c| \cdot f) \Rightarrow$

$\Rightarrow |c| \cdot \text{Sup } f \leq \text{Sup}(|c| \cdot f) \quad (**)$

$\text{a.p } g \text{ no } (*) \text{ k.p } (**)$ l'axioma (vi)

$$\text{vii)} \quad \begin{cases} \inf f \leq f(x) \\ \inf g \leq g(x) \end{cases} \Rightarrow f(x) + g(x) \geq \inf f + \inf g \Rightarrow$$

$$\Rightarrow (\inf f + \inf g) \text{ k\ddot{u}} \text{ zur } (f+g)$$

$$\inf (f+g) \geq \inf f + \inf g.$$

$$\text{viii)} \quad \inf f = \inf ((f+g) - g) = \inf ((f+g) + (-g))$$

$$\stackrel{\text{vii}}{\Rightarrow} \inf (f+g) + \inf (-g) \leq \inf ((f+g) + (-g)) \Rightarrow$$

$$\stackrel{\text{vii}}{\Rightarrow} \inf (f+g) - \sup g \leq \inf f \Rightarrow$$

$$\Rightarrow \inf (f+g) \leq \inf f + \sup g$$

$$\text{ix)} \quad \text{Ano, (vii)} \quad \inf (f+c) \geq \inf f + c$$

$$\text{Ano, (viii)} \quad \inf (f+c) \leq \inf f + c$$

$$\text{z\ddot{u}} \text{g, } \forall x \in A \text{ u (ix)}$$

$$\text{x)} \quad \text{f\ddot{u}} \text{r } c=0, \forall x \in A$$

$$\text{f\ddot{u}} \text{r } c \neq 0, (\forall x \in A): |c| \cdot f(x) \geq |c| \cdot \inf f \Rightarrow$$

$$\Rightarrow |c| \cdot \inf f \leq (|c| \cdot f)(x) \Rightarrow |c| \cdot \inf f \text{ k\ddot{u}} (|c| \cdot f)$$

$$\text{z\ddot{u}} \text{g, } |c| \cdot \inf f \leq \inf (|c| \cdot f) \oplus$$

$$f = \frac{1}{|c|} \cdot |c| \cdot f \Rightarrow \inf f = \inf \left(\frac{1}{|c|} (|c| \cdot f) \right) \oplus$$

$$\stackrel{\oplus}{\Rightarrow} \frac{1}{|c|} \cdot \inf (|c| \cdot f) \leq \inf \left(\frac{1}{|c|} (|c| \cdot f) \right) = \inf f \Rightarrow$$

$$\Rightarrow \inf (|c| \cdot f) \leq |c| \cdot \inf f \oplus \oplus$$

$$\text{z\ddot{u}} \text{g, ano } \oplus \text{ u } \oplus \oplus \text{ } \forall x \in A \text{ u (x)}$$